Lectures 30
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P442 – Analytical Mechanics - II
Forces of Constraint & Lagrange Multipliers

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Generalized Coordinates Revisited

Consider a particle whose position is given by the Cartesian coordinates (x, y, z). If the particle is totally unconstrained then there are three degrees of freedom for the particle and we can choose these to be given by the Cartesian coordinates or other coordinates, such as spherical coordinates (r, θ, ϕ) or cylindrical coordinates (r, ϕ, z) or some other choice – in general (q_1, q_2, q_3). And we can express the Cartesian coordinates in terms of the the generalized coordinates, that is:

$$\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3) \end{aligned} \tag{1}$$

Suppose the particle is constrained to move in a plane, for example, such that $z = 0$. Then there are only two degrees of freedom and one needs only two generalized coordinates. And $z = 0$ is the constraint equation. Or the motion could be constrained to the surface of a sphere of radius R, in which case there are again two degrees of freedom and the constraint equation is $R^2 - (x^2 + y^2 + z^2) = 0$. Now we have:

$$\begin{aligned} x &= x(q_1, q_2) \\ y &= y(q_1, q_2) \\ z &= z(q_1, q_2) \end{aligned} \tag{2}$$

If the particle is free to move only in the $x - y$ plane as a plane pendulum of length ℓ the there are now two constraint equations $z = 0$ and $\ell^2 - (x^2 + y^2) = 0$. Now we have one degree of freedom and:

$$\begin{aligned} x &= x(q) \\ y &= y(q) \\ z &= z(q) \end{aligned} \tag{3}$$

If there are two particles, each totally unconstrained, the system has six degrees of freedom. If, for example, they are connected by a rod so the distance between them is fixed, one of the degrees of freedom is removed.

Holonomic Contraints

If there are N particles then we start with $3N$ Cartesian coordinates among which there may be m constraint equations of the form:

$$f_k(x_i, y_i, z_i, t) = 0 \qquad i = 1, 2, \ldots, N \;\; k = 1, 2, \ldots, m \tag{4}$$

leading to $3N - m$ degrees of freedom. Constraints that are expressible in the form of equation 4 are called *holonomic*. Holonomic constraints are further divided into *rhenomorous* constraints, in which time appears as an explicit variable, and *scleronomous* constraints, in which the constraints are not explicitly dependent on time. The term *holonomic* is also sometimes used to describe a system that has n degrees of freedom and is described by n generalized coordinates.

At the end of this lecture we will discuss two examples of scleronomous constraints – the Atwood's machine and the following yo-yo. Examples of rhenomorous constraints are that of a particle sliding on a ramp where the angle of the incline is increasing with time or a pendulum suspended from the ceiling of an accelerating railroad car.

Nonholonomic Contraints

To show you an example of a nonholonomic constraint consider a vertical disk or radius r rolling in the (x, y) plane as shown in Figure 1. The orientation of the disk can be specified by (θ, ϕ) and the point at which the edge of the disk touches the plane is (x, y). The velocity of the disk in the plane is v. Since we have the contraints $v = r\dot{\phi}$, $\dot{x} = -v\cos\theta$ and $\dot{y} = -v\sin\theta$ you might think that you could integrate these to obtain $x(\theta, \phi)$ and $y(\theta, \phi)$. But you can convince yourself that in general x and y will not return to their starting values after the angles trace out a closed trajectory in $\theta - \phi$ space. This is then an example of nonholonomic constraints. Such constraints are covered in advanced texts.

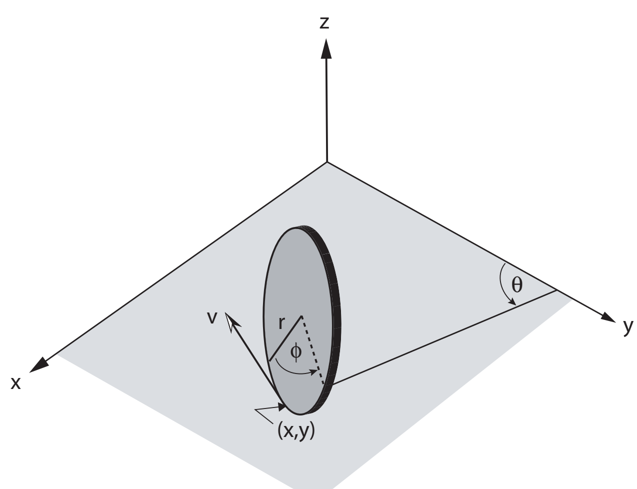

Figure 1: A disk rolling on its edge in the (x, y) plane.

Generalized momenta and forces

Suppose we have a system that is described by generalized coordinates q_j and generalized velocities $\dot{q}_j$ where j goes from 1 to N, the number of degrees of freedom. The N Lagrange's equations are:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} \text{ for } j = 1, N} \tag{5}$$

Compare this to:

$$\frac{dp_j}{dt} = F_j \tag{6}$$

It is tempting to associate $\partial L/\partial \dot{q}_j$ with a generalized momentum and $\partial L/\partial q_j$ with a generalized force – and this indeed works.

Consider a particle of mass m moving in two dimensions under the influence of a force given by the potential energy $U(x, y)$ (see Figure 2. The Lagrangian for the particle is:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y) \tag{7}$$

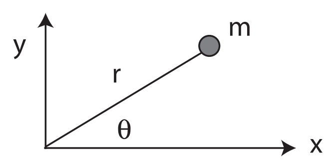

Figure 2: A mass located by x, y or r, θ.

Lagrange's equations for the choice of coordinates x, y yields $d(m\dot{x})/dt = -\partial U/\partial x$ which is the same as $dp_x/dt = F_x$ with a similar equation for the y component.

What happens if we use r, θ instead as our generalized coordinates? Now the Lagrangian is:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r, \theta) \tag{8}$$

First look at the Lagrange's equation for the r coordinate:

$$\frac{d}{dt}m\dot{r} = \frac{dp_r}{dt} = -\frac{\partial U}{\partial r} = F_r \tag{9}$$

and for θ:

$$\frac{d}{dt} m r^2 \dot{\theta} = \frac{dp_\theta}{dt} = -\frac{\partial U}{\partial \theta} \tag{10}$$

You will recognize $p_\theta = mr^2\dot{\theta}$ as being the angular momentum. If U has no θ dependence then $\partial U/\partial\theta = 0$ and angular momentum is conserved. But if $\partial U/\partial\theta \neq 0$ how do we interpret the RHS of equation 10? You need to recall the form of the gradient in polar coordinates:

$$\vec{\nabla} U = \frac{\partial U}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{e}_\theta = -F_r\hat{e}_r - F_\theta\hat{e}_\theta \tag{11}$$

So you see that $-\partial U/\partial\theta = rF_\theta = \tau$ – torque, which is the time rate of change of angular momentum.

One speaks of the *generalized momentum conjugate to the generalized coordinate* q_j as $p_j = \partial L/\partial \dot{q}_j$ and Lagrange's equation is $\dot{p}_j = \partial L/\partial q_j$. If the Lagrangian is independent of q_j the coordinate is said to be *ignorable* and the corresponding generalized momentum is a constant of the motion. In all this we are assuming that the forces are conservative.

Forces of constraint and Lagrange multipliers

In the problems we have been solving so far we have been dealing with holonomic constraints that eliminate one or more degrees of freedom and there are forces associated with these constraints. Forces that we have been ignoring – in fact that is one of the nice features of Lagrange's equations. But sometimes we want to know what the constraining forces are. For this we turn to the technique called Lagrange multipliers.

We will assume a system describable by generalized coordinates q_1 and q_2. Furthermore we suppose that these coordinates are connected by the contraint equation $f(q_1, q_2) = 0$. The variational principle on which our Lagrangian formulation of mechanics is based states that:

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} \right) \delta q_1 + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} \right) \delta q_2 \right] dt = 0 \tag{12}$$

leading to the two Lagrange's equations for our two coordinates. But the coordinates are tied to each other by the constraint equation and $f(q_1, q_2) = 0$ then $\delta f = 0$. But:

$$\delta f = \left(\frac{\partial f}{\partial q_1}\delta q_1 + \frac{\partial f}{\partial q_2}\delta q_2 \right) = 0 \tag{13}$$

So δq_1 and δq_2 are related through equation 13. Using that relation in equation 12 we end up with:

$$\int_{t_1}^{t_2} \left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} \right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} \right) \frac{\partial f/\partial q_1}{\partial f/\partial q_2} \right] \delta q_1 dt = 0 \tag{14}$$

For the above to be satisfied for arbitrary δq_1 the expression in the brackets must vanish which means:

$$\frac{d(\partial L/\partial \dot{q}_1)/dt - \partial L/\partial q_1}{\partial f/\partial q_1} = \frac{d(\partial L/\partial \dot{q}_2)/dt - \partial L/\partial q_2}{\partial f/\partial q_2} = \lambda(t) \tag{15}$$

The reason we added the function $\lambda(t)$ is that the leftmost term depends is a function only of q_1 and the middle term only of q_2. The only way these time dependent terms can be always equal to each other is if they are equal to the same function of time.

With that we have:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \lambda(t)\frac{\partial f}{\partial q_i} \qquad i = 1, 2 \tag{16}$$

The new term in these Lagrange's equations has to be a generalized force – it is in fact the force of constraint. We have two Lagrange's equations and an equation of constraint (three equations) to solve for the three unkowns $q_1(t)$, $q_2(t)$ and $\lambda(t)$ and as a bonus for our hard work we get the forces of constraint. Alternatively we could have just used the equation of constraint to eliminate one of the generalized coordinates and then we would have a single Lagrange's equation without the Lagrange multiplier. But then we would learn nothing about the forces of constraint.

Just to be specific, in our example here, the generalized forces of constraint are Q_i where:

$$Q_i = \lambda(t)\frac{\partial f}{\partial q_i} \qquad i = 1, 2 \tag{17}$$

This is a force if the corresponding generalized coordinate is a spatial coordinate and a torque if the corresponding generalized coordinate is an angular coordinate.

We will work out two simple examples but before we do that here are the results for the more general situation where there are n coordinates and m constraints:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \sum_{j=1}^{m} \lambda_j(t)\frac{\partial f_j}{\partial q_i} \qquad & i = 1, \ldots, n \\ f_j(q_i) = 0 \qquad & j = 1, \ldots, m \\ Q_i = \sum_{j=1}^{m} \lambda_j(t)\frac{\partial f_j}{\partial q_i} \qquad & i = 1, \ldots, n \end{aligned} \tag{18}$$

So here we have n unknown functions for the generalized coordinates $q_i(t)$ and m unknown multipliers $\lambda_j(t)$. To find these we have on hand $n+m$ equations: n Lagrange's equations, one for each generalized coordinate, and m constraint equations $f_j(q_i) = 0$. And from this we get n generalized forces of constraint Q_i.

Atwood's machine Let's re-visit Atwood's machine as in Figure 3 (a). The Lagrangian is:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + Mgx + mgy \tag{19}$$

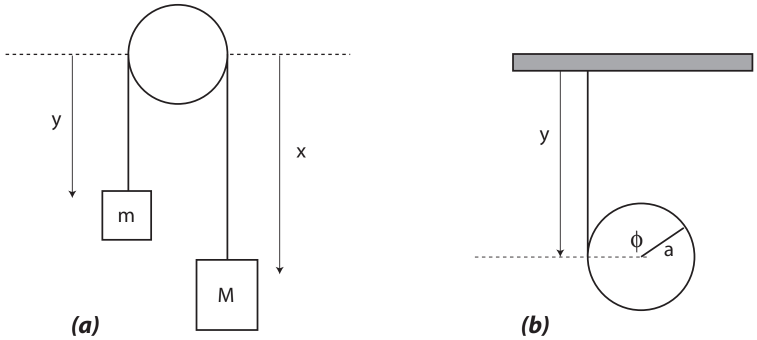

Figure 3: (a) Atwood's machine and (b) falling yo-yo.

and the equation of constraint is $f(x, y) = x + y =$ constant. Lagrange's equations with multipliers:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} &= \frac{\partial L}{\partial y} + \lambda \frac{\partial f}{\partial y} \end{aligned} \tag{20}$$

which leads to the equations of motion:

$$\begin{aligned} M\ddot{x} &= Mg + \lambda \\ m\ddot{y} &= mg + \lambda \end{aligned} \tag{21}$$

You will recognize that $\lambda = -T$ where T is the tension in the string.

Falling yo-yo This is shown in Figure 3 (b). The Lagrangian and equation of constraint are:

$$\begin{aligned} L = \frac{1}{2}m\dot{y}^2 + \frac{1}{4}ma^2\dot{\phi}^2 + mgy \\ f(y, \phi) = y - a\phi = 0 \end{aligned} \tag{22}$$

The mass of the disk is m and the moment of inertia is $ma^2/2$. As in the previous example apply Lagrange's equations with the Lagrange multiplier and you will find:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= \frac{\partial L}{\partial \phi} + \lambda \frac{\partial f}{\partial \phi} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} &= \frac{\partial L}{\partial y} + \lambda \frac{\partial f}{\partial y} \end{aligned} \tag{23}$$

leading to:

$$
\begin{aligned}
&\frac{1}{2}ma^2\ddot{\phi} = -\lambda \\
&m\ddot{y} = mg + \lambda
\end{aligned}
\tag{24}
$$

Solving you have: $\lambda = -mg/3$. The general constraint forces are $Q_y = -mg/3$ and $Q_\phi = mga/3$. The latter is actually a torque.