Lecture 21
Friday - October 21, 2005
Written or last updated: October 21, 2005

P441 – Analytical Mechanics - I
Roche Limit – Tides

Roche Limit

Suppose a satellite is orbiting a planet – for example the Moon (mass m) orbiting the Earth (mass M) separated by distance d. There is a limit to how close the the satellite can approach the planet without having the satellite breaking up due to tidal forces. The assumption is that astronomical bodies are held together by self-gravity. So let's try this out for the Earth-Moon system. We will make the following simplifying assumption: that the two hemispheres of the moon can be treated by two spheres separated by r, the radius of the Moon. Please see Figure 1. The attractive force between these two spheres is:

$$F_a = \frac{G(m/2)^2}{r^2} \tag{1}$$

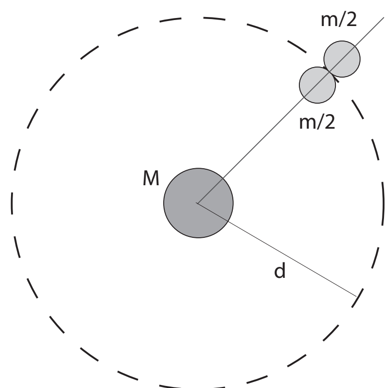

Figure 1: The Moon orbiting the Earth where the Earth and Moon are separated by distance d. We model the moon as two spheres, each of mass $m/2$ whose centers are separated by r.

The tidal force trying to rip the two spheres apart is F_t:

$$F_t = \frac{GMm}{2}\left(\frac{1}{(d-r/2)^2} - \frac{1}{(d+r/2)^2}\right) = \frac{GMm}{2d^2}\left(\frac{1}{(1-r/2d)^2} - \frac{1}{(1+r/2d)^2}\right) \tag{2}$$

If we assume $d >> r$ and use the binomial expansion we end up with this:

$$F_t \approx \frac{GMm}{d^3} r \tag{3}$$

Setting $F_a = F_t$ we get the Roche limit:

$$d = \left(\frac{4M}{m}\right)^{1/3} r \tag{4}$$

If we assume that $M = 4\pi R^3 \rho_E/3$ (R is the Earth radius and ρ_E the density) and similarly $m = 4\pi r^3 \rho_M/3$ for the moon then the Roche limit is:

$$d = \left(\frac{4\rho_E}{\rho_M}\right)^{1/3} R \tag{5}$$

It happens that $\rho_M/\rho_E = 3/5$ so this gives $d \approx 7,500$ miles. The actual answer is $d \approx 10,000$ miles.

Tides

We will discuss tides caused by the Moon on the Earth[1]. The Earth and Moon rotate about their common center-of-mass (CM) which is located inside the Earth about 3000 miles from the center of the Earth as shown in Figure 2.

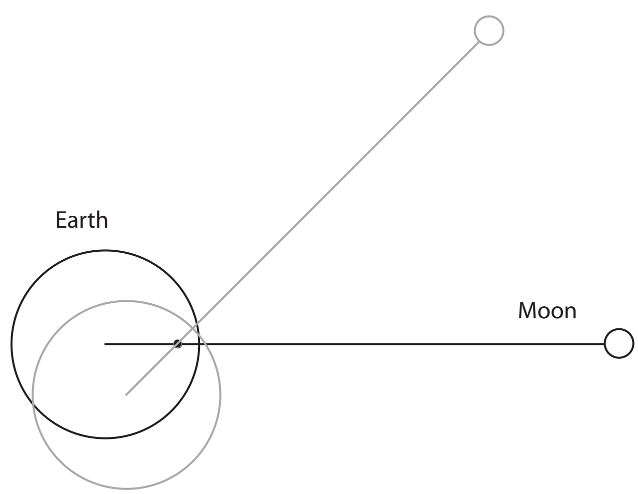

Figure 2: The Earth and Moon rotate about their common center-of-mass (CM) located inside the Earth and 3000 miles from the center of the Earth.

With respect to the CM the Earth's center-of-mass has an acceleration a_C given by:

$$a_C = \frac{GM_m}{r_m^2} \tag{6}$$

where M_m is the Moon's mass and r_m is the Earth-Moon distance. For a particle of mass m on the Earth or in the Earth, the particle experiences a centrifugal force given by ma_C which is superposed on any other force that the particle experiences.

[1]Much of this discussion is taken from the text *Newtonian Mechanics* by A. P. French in the M.I.T. Introductory Physics Series - see pg 532 ff

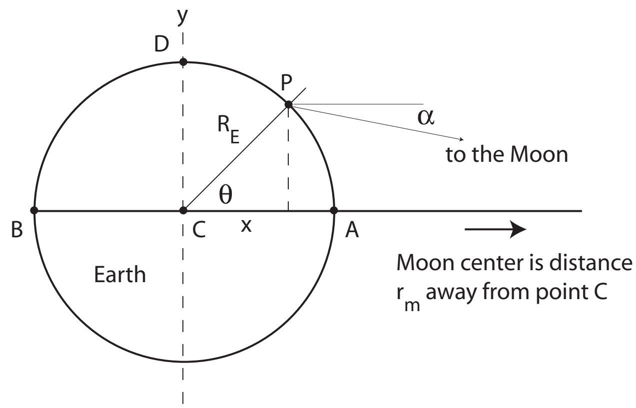

Figure 3: The Earth and Moon rotate about their common center-of-mass (CM) located inside the Earth and 3000 miles from the center of the Earth.

Just a note on Figure 2 – in this picture the Earth and Moon always present the same face to each other. In addition the Earth rotates.

For a particle at the Earth's center, the centrifugal force and the Moon's force of gravitational attraction are equal and opposite. If that same particle is on the surface of the Earth at point A (see Figure 3) the gravitational force is greater than the centrifugal force by an amount f_0 and pointing towards the Moon. And:

$$f_0 = \frac{GM_m m}{(r_m - R_E)^2} - \frac{GM_m m}{r_m^2} \tag{7}$$

We can re-write this as:

$$f_0 = \frac{GM_m m}{r_m^2} \left[\left(1 - \frac{R_E}{r_m} \right)^{-2} - 1 \right] \tag{8}$$

Since $R_E \approx r_m/60$ we can assume that $R_E << r_m$ and make this approximation:

$$f_0 \approx \frac{2GM_M m}{r_m^3} R_E \tag{9}$$

Similarly, at point B (see Figure 3) the particle experiences a force f_0 directed away from the Moon.

Now consider a point P on the Earth's surface located by angle θ or $x = R_E \cos\theta$ and $y = R_E \sin\theta$. Equation 9 becomes:

$$f_x = \frac{2GM_M m}{r_m^3} x = \frac{2GM_M m}{r_m^3} R_E \cos\theta \tag{10}$$

But there is also a transverse force given by:

$$f_y = -\frac{GM_M m}{r_m^2}\sin\alpha \tag{11}$$

where we assume that the distance between point P and the center of the Moon is approximately r_m. We also have:

$$\tan\alpha = \frac{y}{r_m - x} \tag{12}$$

The largest value α can take on is $\tan^{-1} R_E/r_m \approx\ 1°$, so we can approximate $\sin\alpha \approx \tan\alpha$ and $\tan\alpha \approx y/r_m$ giving us:

$$f_y = -\frac{GM_M m}{r_m^3} R_E \sin\theta \tag{13}$$

To find the height of the tide, consider the work done by the tidal forces in moving a particle of water from point D to point A. This should be equal to the gravitational potential energy needed to raise the particle a height h against the Earth's gravitational pull. First the work:

$$dW = f_x dx + f_y dy = \frac{GM_m m}{r_m^3}(2xdx - ydy) \tag{14}$$

and

$$W_{DA} = \frac{GM_m m}{r_m^3}\left[\int_0^{R_E} 2xdx - \int_{R_E}^0 ydy\right] = \frac{3GM_m m}{2r_m^3}R_E^2 \tag{15}$$

Set this equal to mgh:

$$h = \frac{3GM_m R_E^2}{2gr_m^3} \tag{16}$$

yielding $h \approx 0.54$ m. This is for the open sea. We assume in SI units: $G = 6.67 \times 10^{-11}$, $M_m = 7.34 \times 10^{22}$, $r_m = 3.84 \times 10^8$, $R_E = 6.37 \times 10^6$ and $g = 9.8$.

What about the effect of the Sun? In SI units $M_s = 1.99 \times 10^{30}$ and $r_s = 1.49 \times 10^{11}$ and that means for a particle on the Earth the ratio of the gravitational forces due to the Sun and Moon is $F_s/F_m \approx 180$. But what matters is the gradient of the force – which is why the Moon wins out.

The gravitational force is given by $F(r) = GMm/r^2$ and:

$$f = \Delta F = -\frac{2GMm}{r^3}\Delta r \tag{17}$$

For the Moon we make the substitutions $M = M_m$, $r = r_m$ and $\Delta r = \pm R_E$ to get $f = \pm f_0$. And the ratio of tide producing forces are given by:

$$\frac{f_s}{f_m} = \frac{M_s}{M_m}\left(\frac{r_m}{r_s}\right)^3 \approx \; 0.465 \tag{18}$$