P360

P360: Physical Optics
Supplementary Note # 1: Principle of Least Time

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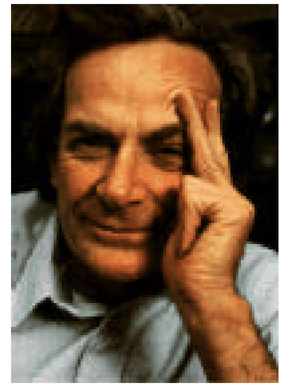
Geometric Optics and Fermat's Principle

A nice way to approach the subject of geometric optics is by applying Fermat's Principle of Least Time (FPLT). There is a nice discussion of this by Feynman in his *Lectures on Physics*[1]. In a later book [2], in which Feynman lays out the principles of quantum electrodynamics in a series of lectures for non-experts, he shows why FPLT works.



Richard P. Feynman

The statement of the principle is pretty simple: when light travels from point A to point B it will follow the path which takes the least time. Combined with the possibility that the index of refraction of the medium through which the light travels may be varying, FPLT explains a number of phenomena. Let's take two examples.

Suppose you are looking out over a long highway on nice hot and sunny day. In the distance the road appears to be 'wet.' This mirage can be explained by FPLT. The light from the sky above can reach your eye by a straight line but if the road is hot the index of refraction of the air close to surface is less than the index of refraction in the air above so the path followed by the light rays actually is bent, as shown in Figure 1. Where the index of refraction, n, is lower, the velocity of light

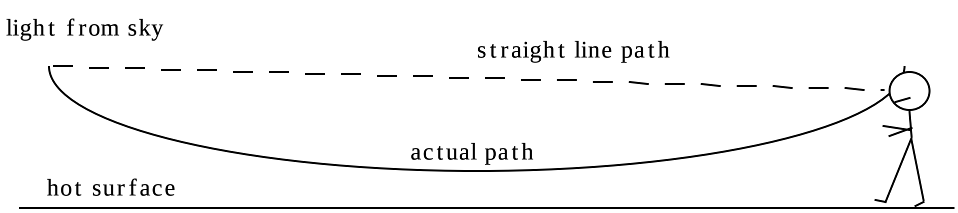


Figure 1 - A mirage: a road on a hot day appears to be wet in the distance. This phenomenon can be understood from Fermat's Principle of Least Time.

in the air, c/n, is greater. The transit time along the curved path is less than for the straight line path. Your brain interprets this phenomenon as light entering your eye coming from the reflection of light from sky off a wet surface in the distance.

Another phenomenon is associated with the setting sun. When see the sun as it is just about to disappear over the horizon, it already is below the horizon. You see the sun since the light obviously does not follow a straight line path to your eyes, rather the light comes along a path which minimizes travel time, spending time up high where the index of refraction is near one, before reaching your eyes. You interpret this as light coming from a sun which is still above the horizon.

We will now use FPLT to derive the law of reflection and Snell's Law - two cornerstones of the geometric optics. We will also use the principle to understand focusing properties of mirrors and lenses.

Law of Reflection

Figure 2 two shows point A as a source of light and point B as the receiver. Assuming that the index of refraction is everywhere the same, the minimum time path will be a straight line between A and B, but if we require that the light first reflect off the mirror as shown in the figure, what path will the light take? One possibility is for the light to travel straight down from point A to the mirror and then travel in a straight line to point B (path 1). Yet another possibility is for the light to head to a point on the mirror just below point B and then straight up to point B (path 2). The travel time for paths in which the light bounces off the mirror to the left of A or to the right of B would of course be longer. Is there a point on the mirror between points a and b which at which the light bounces for which the travel time is a minimum? Yes there is and to find it consider Figure 2. The time it takes for light to travel from point A to the mirror to point B is just:

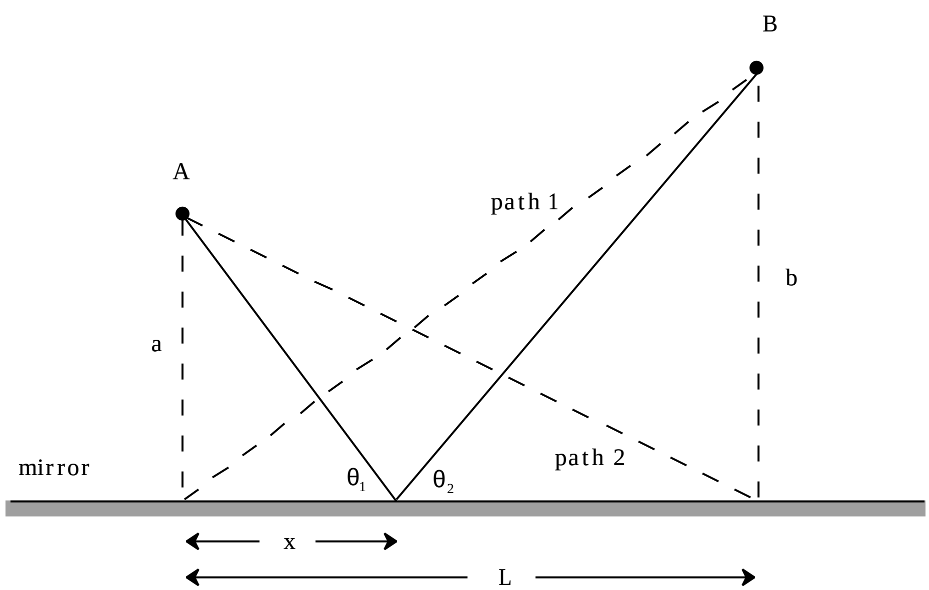


Figure 2 - Light travels from point A to point b by bouncing off a mirror. FPLT tells us that the angles of incidence and reflection are equal.

$$t(x) = \frac{1}{c}\left(\sqrt{a^2 + x^2} + \sqrt{b^2 + (L - x)^2}\right)$$

Setting dt/dx = 0 results in:

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{(L - x)}{\sqrt{b^2 + (L - x)^2}}$$

which is equivalent to requiring $\cos\theta_1 = \cos\theta_2$ or $\theta_1 = \theta_2$ - the angles of incidence and reflection are equal. If we plot the time as a function of x, for the case that a = b = 1 and L = 2 we get the curve of Figure 3. Not surprising, the minimum occurs when x = L/2 which corresponds to the incident and reflected angles being equal. In Chapter 26 or reference 1, Feynman gives a neat geometric proof of this result.

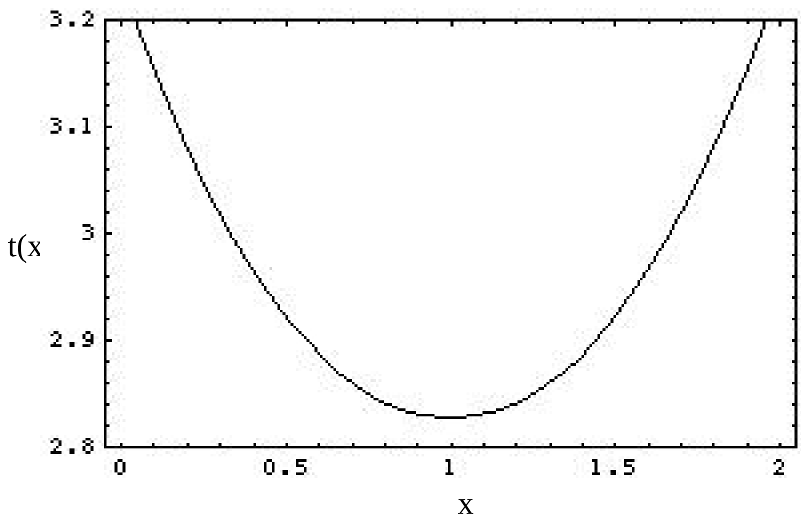


Figure 3 - Light travel time, t(x), as a funtion of x for the case shown in Figure 2 with a = b = 1 and L = 2.

Snell’s Law

Now consider the situation shown in Figure 4 where light leaves point A and arrives at point B. The difference now is that point A is in a medium with index of refraction equal to one and point B is in a medium with index of refraction n. As before we find the travel time:

$$t(x) = \frac{1}{c}\left(\sqrt{a^2 + x^2} + n\sqrt{b^2 + (L - x)^2}\right)$$

Imposing the requirement dt/dx = 0 gives us:

$$\frac{x}{\sqrt{a^2 + x^2}} = n \frac{(L - x)}{\sqrt{b^2 + (L - x)^2}}$$

or

$$\cos\theta_1 = n \ \cos\theta_2$$

and since $\cos\theta_1 = \sin\theta_i$ and $\cos\theta_2 = \sin\theta_r$ we arrive at Snell's Law:

$$\sin\theta_1 = n \ \sin\theta_2$$

Clearly the strategy to minimize travel time is to stay in the medium with the lower index of refraction as long as possible before plunging into the medium with the higher n. A straight line shot from point A to point B will not work. Those photons are smart.

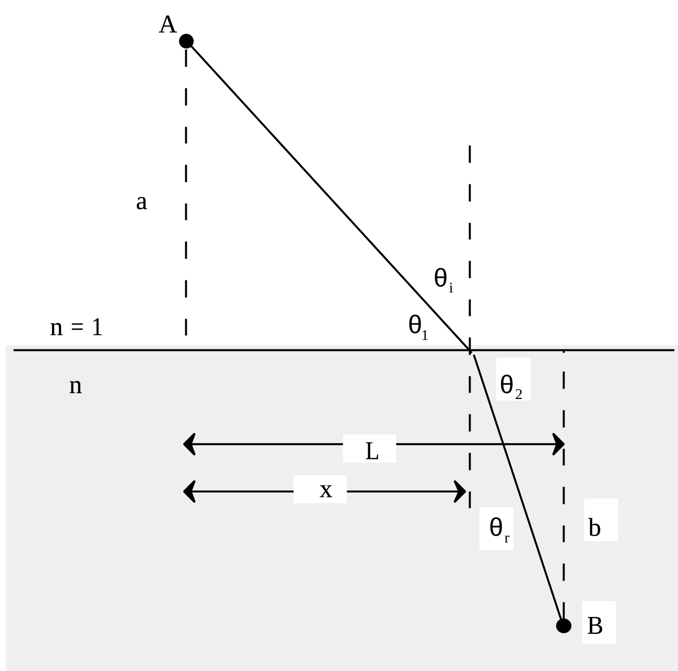


Figure 4 - Light travels from point A to point B, passing from a medium to another with different indices of refraction. FPLT leads to Snell's Law.°

Focusing Light with Mirrors

A lot of what we will be doing in geometric optics is focusing light. We will be taking *rays* of light from a source and focusing those rays - bringing them together - somewhere else and we'll exploit reflection and refraction to do this.

Before we move on it is useful to point out that we can have reflection when light is incident on a mirror or on an interface between two media of differing indices of refraction. In the latter case we have both reflection and refraction and we will be discussing this in more detail later. We will see, for example, that the amount of reflection depends on the polarization of the light. We will also discuss techniques for eliminating reflection - but more on that later.

Let us look at the reflection of light from a spherical mirror as we show in Figure 5. The center of the spherical mirror is at the point O and the radius is R. The dashed line through O and O' is the axis of the mirror. A ray of light starts from point P and reflects off the mirror at point M. The angles of incidence and reflection are equal and the reflected ray intersects the axis at point O'. Note that $\phi = 2\theta$ and as long as the angles are small, the distance between O and O' is R/2. If we have a collection of parallel rays at varying distances from the axis of the mirror, as long at the angles are small, all the rays would be focused at the same point O' and the focus is a distance R/2 from the mirror. Light from a very distant source would strike the mirror as a collection of parallel rays and focus at point O'. Alternatively, a source of light at O' would reflect off the mirror to produce a parallel beam. This we obtain from the law of reflection. However this only works for rays which are close to the axis of the mirror so the angles involved are all small.

Let's now consider Figure 6 where we plot a parabola with focus at x = 1 and y = 0 and a directrix (line) defined by x = -1. Recall that a parabola is the locus of all points equidistant from a point (focus) and a line (directrix). The axis of the parabola passes through the focus and is perpendicular to the directrix. The equation for our particular parabola is:

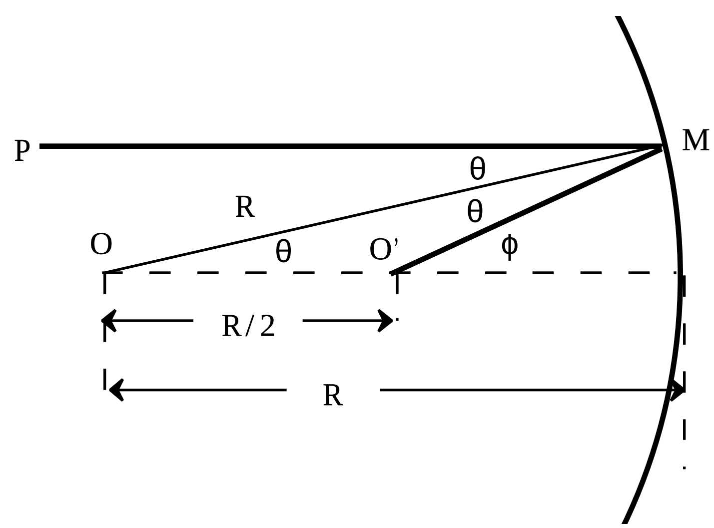


Figure 5 - Light ray traveling parallel to the axis of a spherical mirror reflects off the mirror and passes through the focus of the mirror located a distance R/2 from the mirror where R is the radius of the mirror.

$$y^2 = 4x$$

We also draw a circle of radius R where R = 2 and the center of the circle lies on the axis of the parabola at x = 2 and y = 0. The equation of the circle is given by:

$$y^2 = 4x - x^2$$

If the focus of the parabola is at $(f, 0)$ with vertex at the origin and the center of the circle at $(R,0)$ then we have for the parabola:

$$y^2 = 4fx$$

and for the circle:

$$y^2 = 2Rx - x^2$$

Notice that in the region near the axis of the parabola, the two curves approximate one another. That is not surprising since the difference in y values is x^2.

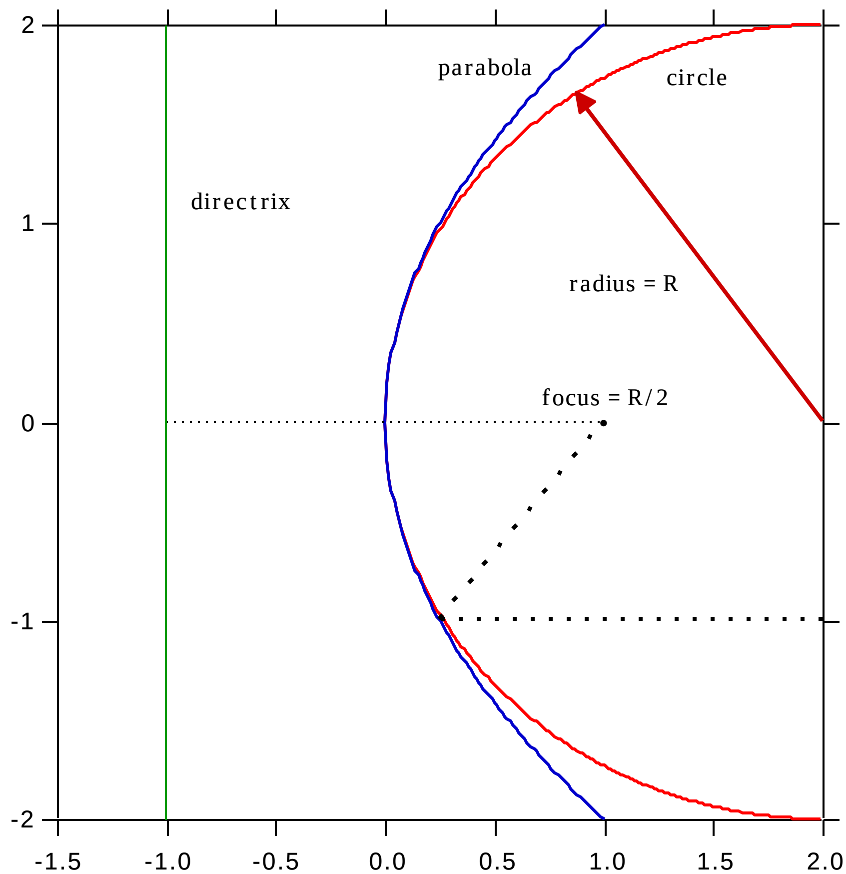


Figure 6 - A circle and parabola are plotted. The center of the circle is on the x-axis at x = R = 2 and the focus of the parabola is also on the x-axis at x = R/2. The radius of the circle is R and the directrix is at x = -R/2.

Now consider the vertical line drawn by x = 1.5, for example. This line is parallel to the directrix and let's call the distance between the two lines c. The distance between the directrix and the parabola is equal to the distance from the parabola and the focus and we call that a. Notice then the any path from our line at x = 1.5 to the parabola (parallel to the axis of the parabola) and then to the focus has a total path length of c (see the dashed line). Suppose parallel rays of light, say from a distant source, all arrive at the line at x = 1.5. These rays are also perpendicular to the line. All the rays are focused at the focus of the parabola because they all take the same time to travel. There is no minimum path since all the path lengths we are considering have the same length. This is the condition for a focus. It is exactly true for a parabolic mirror and only approximately true for a spherical mirror as long as stay close to the axis of the mirror so the angles are small. Obviously it does not matter where the line is located - it just has to be parallel to the directrix.

An ellipse is the locus of points such that the total distance between one focus of the ellipse to a point on the ellipse and then to the other focus is a constant. So if we fashioned a mirror in the shape of an ellipsoid with the reflective part on the inside and had light rays emerging from one focus they would all converge at the other focus.

Liquid Mirror Telescopes

Spherical mirrors are easier to make then parabolic mirrors but they have the problem of imperfect focusing as pointed out above. Mirrors used in telescopes are spherical. One way to make a parabolic mirror is to spin a heavy liquid, like mercury. A rotating fluid takes the shape of a paraboloid. The focal length, f, of such a mirror is related to the acceleration of gravity, g, and the angular velocity, ω, with which the liquid is spinning:

$$f = \frac{g}{2\omega^2}$$

Telescopes using such mirrors have been built and are being built. Obviously the mirror can only point up and there are safety considerations. The Large Zenith Telescope will be built near Vancouver, CA. It will have 28 liters of mercury spinning in a large pan at one revolution per 8.5 sec. See the websites:

http://www.astro.ubc.ca/LMT/lzt.html

and

http://wood.phy.ulaval.ca/lmt/home.html

1 Feynman, Leighton and Sands, *Lectures on Physics*, Vol I, Addison-Wesley, Reading MA (1963) Chapters 26 and 27.

2 Feynman, R. P., *QED: The Strange Theory of Light and Matter*, Princeton Press, Princeton, NJ (1985).