A. C. Circuits - Part II

Honor Physics Note 005
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Honors Physics P222 - Spring, 2004

Introduction

In this note we discuss electrical oscillating circuits: undamped, damped and driven.

LC oscillator - undamped

FIgure 1(a) shows a capacitor C and an inductor L in series with a switch S. The impedance across the series combination of the C and L is given by:

$$\hat{Z}_{eq} = i\left(\omega L - \frac{1}{\omega C}\right) \tag{1}$$

If $\omega = 1/\sqrt{LC}$ this impedance is zero – a current can flow without an external voltage source - assuming that the capacitor is initially charged. In fact if the capacitor is charged with charge Q and the switch is closed the capacitor discharges through the inductor and the changing current through the inductor leads to a voltage across the inductor that changes with time and is always equal to the voltage across the capacitor. Kirchoff's voltage law leads to:

$$L\frac{dI}{dt} + \frac{Q}{C} = 0 \tag{2}$$

Using $I = dQ/dt$ and re-arrangement yields:

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q \tag{3}$$

This means that the charge on the capacitor oscillates: $Q(t) = Q_0 \cos\omega t$ where $\omega = 1/\sqrt{LC}$. But since $V = Q/C$ the voltage across the capacitor oscillates as does the current in the inductor. This is reminiscent of the equation for the displacement x of the mass m from equilibrium in a spring-mass system where the spring constant is k:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{4}$$

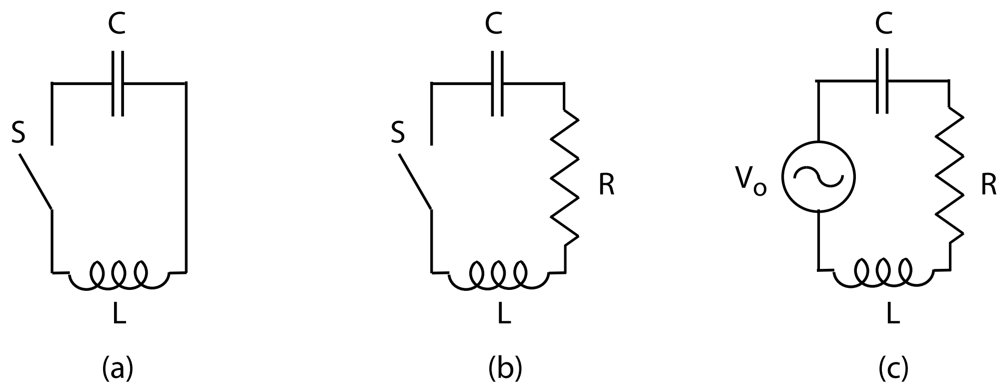


Figure 1: (a) If the capacitor is initially uncharged and the switch is closed, the circuit will oscillate with frequency $\omega = 1/\sqrt{LC}$; (b) If a resistor is added to the circuit oscillation still occurs but with a current amplitude that decays exponentially with time; (c) A driven RLC circuit.

The circuit of Figure 1(a) is the electrical analog of the spring-mass system where L replaces m and $1/C$ replaces k. When the charge on the capacitor is a maximum the current through the inductor is zero and vice-versa – this because the current and charge are out of phase by 90^o. For a mechanical oscillator, when the displacement is a maximum the velocity is zero and vice-versa.

In a mechanical oscillator the total mechanical energy is the sum of the kinetic energy, $mv^2/2$ plus potential $kx^2/2$. For the electrical oscillator the total electrical energy is the sum of the energy stored in inductor $LI^2/2$ plus the energy in the capacitor $Q^2/2C$. In both cases, when one type of energy is a maximum the other is zero.

RLC oscillator - damped

Figure 1(b) shows the same circuit with as in Figure 1(a) but now with a resistor in the circuit. If the capacitor is initially charged and the switch is closed at $t = 0$ oscillation still occurs but eventually the oscillations die down – the initially stored energy is eventually all dissipated as heat in the resistor. Applying Kirchoff's voltage law to the circuit of Figure 1(b) implies the following modification of equation 2:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = 0 \tag{5}$$

Using $I = dQ/dt$ and re-arrangement yields:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \tag{6}$$

This looks similar to the equation of motion for a mechanical oscillator where the damping force is proportional to velocity with damping factor b:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{7}$$

We will solve equation 7 by assuming that the solution must take into account both the damping and an oscillation. Why? We already saw from the above discussion what without a resistor we expect an oscillation. We saw from our discussion of RC and LR circuits that we expect exponential damping. If we just have oscillation without a resistance we expect $Q = Q_0 e^{i\omega t}$. To include damping we simply make the replacement $\omega \rightarrow \omega + i\alpha$ where α is some real number. Then the solution looks like $Q = Q_0 e^{i(\omega+i\alpha)t} = Q_0 e^{-\alpha t} e^{i\omega t}$. Thus we have oscillation with an amplitude that decays away exponentially with time.

To substitute our trial solution into equation 7 we need to calculate the first and second derivatives. For the first derivative:

$$\frac{dQ}{dt} = i(\omega + i\alpha)Q \tag{8}$$

and for the second derivative:

$$\frac{d^2Q}{dt^2} = -(\omega + i\alpha)^2 Q \tag{9}$$

Substitution into equation 5 yields:

$$-L(\omega + i\alpha)^2 Q + iR(\omega + i\alpha)Q + \frac{Q}{C} = 0 \tag{10}$$

where it should be clear that in the above $Q = Q_0 e^{i(\omega+i\alpha)t}$. But the point is that we cancel Q in the above and we are left with an algebraic equation – our differential equation has become an algebraic equation.

$$-(\omega + i\alpha)^2 + i\frac{R}{L}(\omega + i\alpha) + \frac{1}{LC} = 0 \tag{11}$$

Actually equation 11 is really two equations because it involve complex numbers. We will leave it as an exercise for you to show that the solution that this yields has:

$$\alpha = \frac{R}{2L} \tag{12}$$

and

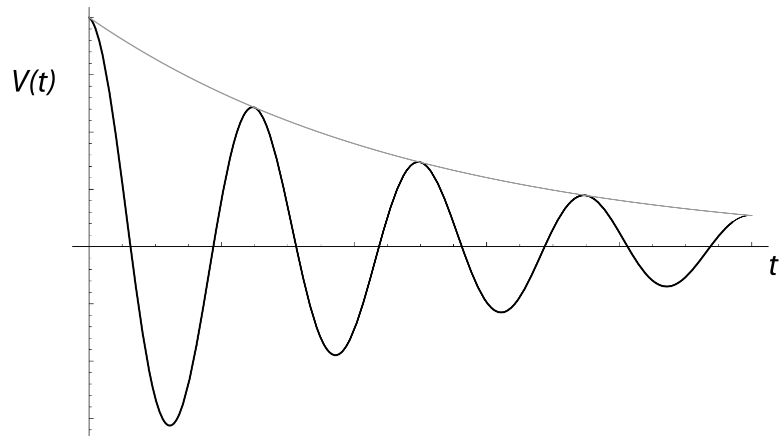


Figure 2: Plot of the dependence of voltage across the capacitor as a function of time for the damped RLC circuit of Figure 1(b) assuming the capacitor is fully charged at $t = 0$. The black curve corresponds to equation 14 and the gray curve shows the exponential decay of the amplitude.

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}} \tag{13}$$

If we assume that the capacitor is fully charged at $t = 0$, and recalling that $V = Q/C$, the voltage across the capacitor is:

$$V_C(t) = \frac{Q_0}{C} e^{-Rt/2L} \cos \omega t \tag{14}$$

In Figure 2 the solution of equation 14 is plotted as a function of time. You can convince yourself, by studying equations 12, 13 and 14 or by experimenting with a plotting program (like *Mathematica*) that when several oscillations of the solution are observed before the solution dies away, then the damped frequency ω is close to the natural frequency ω_0 – or $\omega_0^2 >> R^2/4L^2$.

From equation 13 we can also see that if $\omega_0^2 < R^2/4L^2$ there will be no oscillations.

RLC oscillator - driven

Figure 1(c) shows the circuit of Figure 1(b) with the switch replace by an a.c. voltage source. We assume that the frequency of this driving voltage is ω_d and its amplitude is V_d so equation 7 is modified by the inclusion of this driving voltage on the right-hand-side:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_d e^{i\omega_d t} \tag{15}$$

What is the solution to the above? We know that the solution is the sum of two solution, the so-called *homogeneous* solution which is the solution with the right-hand-side set equal to zero – this is just the solution we found for equation 7 – and the so-called *particular* solution with the right-hand-side non-zero. We know that if we wait long enough the homogeneous solution will decay away so all that is left is the particular solution. We would expect that the particular solution should have the same frequency as the driving frequency. So let's assume a solution of the form $Q = Q_d e^{i\omega_d t}$.

Substituting into equation 15 and after a little re-arranging:

$$\left(-\omega_d^2 + 2i\alpha\omega_d + \omega_0^2\right) Q_d e^{i\omega_d t} = \frac{V_d}{L} e^{i\omega_d t} \tag{16}$$

where $\alpha = R/2L$ and $\omega_0 = 1/LC$. We can cancel the $e^{i\omega_d t}$ and divide by C so that the left-side is the voltage across the capacitor. The voltage across the capacitor will oscillate with frequency ω_d and will have an amplitude that is a function of this driving frequency: $V_C(t) = V_{C0}(\omega_d)e^{i\omega_d t}$. From equation 16:

$$V_{C0}(\omega_d) = \frac{\omega_0^2 V_d}{\left(-\omega_d^2 + 2i\alpha\omega_d + \omega_0^2\right)} \tag{17}$$

Using the usual re-write for our complex quantity in the denominator we have:

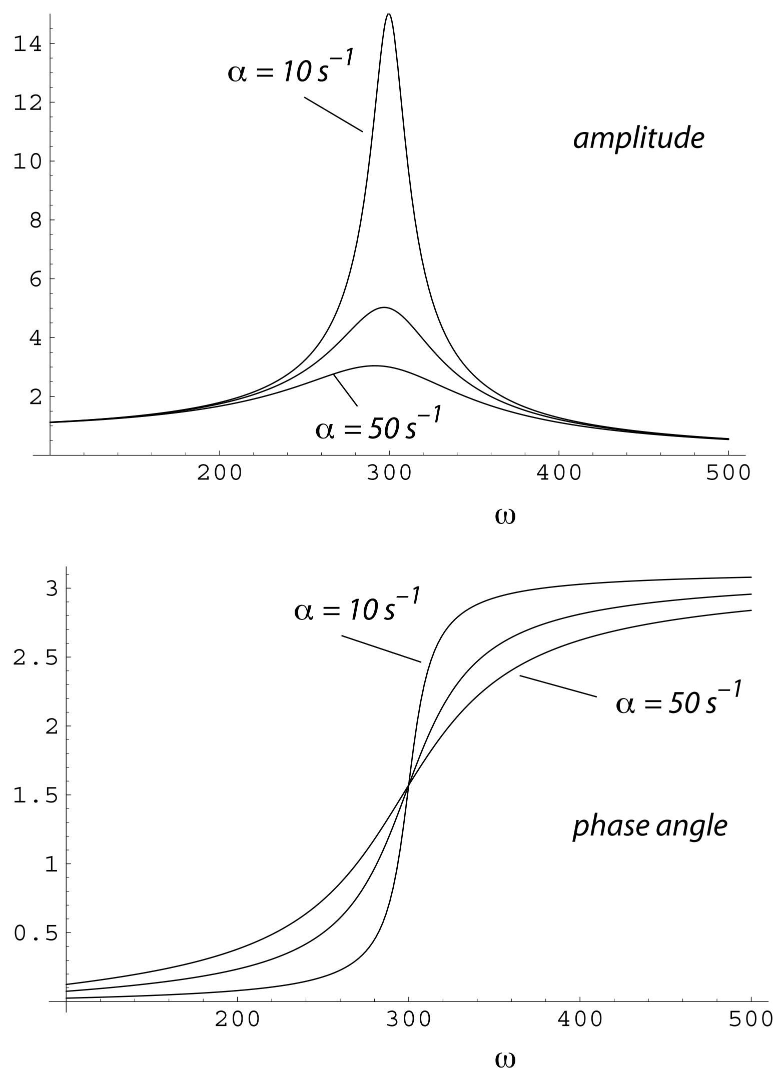
$$V_{C0}(\omega_d) = \frac{\omega_0^2 V_d e^{i\phi}}{\sqrt{(\omega_d^2 - \omega_0^2)^2 + 4\alpha^2\omega_d^2}} \tag{18}$$

where:

$$\phi = \tan^{-1} \frac{2\alpha\omega_d}{\omega_0^2 - \omega_d^2} \tag{19}$$

So the amplitude of the voltage across the capacitor is a maximum when the driving frequency is equal to the resonant frequency. And there is a phase angle ϕ between the voltage across the capacitor and the driving voltage. That angle is $\pi/2$ at the resonant frequency.

Figure 3 shows a plot of the voltage amplitude (given by equation 18) with $\omega_0 = 300\ s^{-1}$ of the driven oscillator as a function of driving frequency for three different values of α of 10 s^{-1}, 30 s^{-1} and 50 s^{-1}. Also shown is the phase angle.



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